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A generalisation of Cartan's theory of gravitation†

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Abstract. A general spinor interaction with gravitation which includes both Einstein's and Cartan's theories as special cases is discussed. The coupling of torsion to matter field in the new theory has an arbitrary strength which can only be determined by experiment.

Soon after Einstein proposed his theory of gravitation based on Riemannian metric of space-time Cartan (1922, 1923, 1924) suggested a generalisation of the theory to include torsion as well by noticing that the tangent space of each space-time point admits not just Lorentz group but the full Poincaré group as its symmetry. Thus in Cartan's geometry a vector, after Cartan displacement along a closed path made of a basis e_i and e_j, gets not only rotated but also translated

$$\delta_{ij}e_k = R_{ijk}{}^l e_l - T_{ij}{}^l e_l \tag{1}$$

where $R_{ijk}{}^l$ and $T_{ij}{}^k$ are the curvature and the torsion tensors repectively. This way the geometry exhibits the full local Poincaré invariance rather than just Lorentz group as the holonomy group.

Cartan's idea was revived by Weyl (1950), and made more precise as a gauge theory of Poincaré group by Kibble (1961) and Sciama (1962). In the gauge formalism of Cartan's theory the gauge potentials of the translational and Lorentz subgroups play the role of the vierbein and the affine connection respectively (Cho 1976a, Heyl *et al* 1976 and references therein). The difference between Cartan's and Einstein's theories is of course that the former has a spin-spin contact interaction whose coupling strength is uniquely fixed in terms of the gravitational constant, whereas the latter simply does not have that coupling. In this note, however, we would like to show that there exists a more general theory which allows an arbitrary coupling strength for the spin-spin interaction and thus yields both Einstein's and Cartan's theories as special cases of the general theory.

Let us start from Cartan's theory. In a local orthogonal frame ∂_i ‡ satisfying

$$[\partial_i, \partial_j] = f_{ij}{}^k\, \partial_k \tag{1}$$

the theory may be described by the Lagrangian

$$\mathscr{L} = (-g)^{1/2}[\tfrac{1}{2}\mathrm{i}(\bar{\psi}\gamma^i D_i\psi - \overline{D_i\psi}\gamma_i\psi) - m\bar{\psi}\psi + (1/\kappa^2)R_{ij}{}^{ij}] \tag{3}$$

where D_i is the covariant derivative of the Lorentz gauge group

$$D_i = \partial_i + \tfrac{1}{2}C_i{}^{jk}S_{jk}; \qquad S_{jk} = -\tfrac{1}{4}[\gamma_j, \gamma_k], \tag{4}$$

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‡ For the notation see Cho (1976a).

$C_i^{jk} = -C_i^{kj}$ being the gauge potentials, and

$$R_{ij}{}^{kl} = \partial_i C_j^{kl} - \partial_j C_i^{kl} + C_i^{km} C_j^{lm} - C_j^{km} C_i^{lm} - f_{ij}{}^m C_m{}^{kl} \tag{5}$$

its field strength. Note that the field C_i^{jk} is not dynamically independent and can be replaced by

$$C_{ijk} = C_{ijk}^{(0)} + C_{ijk}^{(1)} \tag{6}$$

where

$$C_{ijk}^{(0)} = \tfrac{1}{2}(f_{ijk} - f_{ikj} - f_{jki}) \qquad C_{ijk}^{(1)} = \tfrac{1}{8}\kappa^2 \epsilon_{ijkl} \bar{\psi}\gamma^l \gamma^5 \psi.$$

The fields $C_{ijk}^{(1)}$ are called the contortion tensors. Using equation (6), the Lagrangian (3) can be rewritten in terms of dynamically independent variables alone:

$$\mathscr{L} = (-g)^{1/2}[\tfrac{1}{2}\mathrm{i}(\bar{\psi}\gamma^i\, \partial_i\psi - \partial_i\bar{\psi}\gamma^i\psi) - m\bar{\psi}\psi - (1/8)\epsilon^{ijkl} f_{ijk}\bar{\psi}\gamma_l\gamma_5\psi$$
$$+\tfrac{3}{32}\kappa^2(\bar{\psi}\gamma^i\gamma^5\psi)^2 + (1/\kappa^2)R_E] \tag{3'}$$

where R_E is the Einstein curvature i.e. the scalar curvature obtained by the metric *alone*. The repulsive spin–spin interaction term is of course the characteristic of Cartan's theory which is absent in Einstein's theory.

Now we will generalise the theory further so that one can have an arbitrary coupling strength for the spin–spin interaction. To do this it is crucial to realise that under a local Lorentz transformation both f_{ijk} and the gauge potential C_{ijk} transform inhomogeneously, but the torsion T_{ijk} given by

$$T_{ijk} = C_{ijk} - C_{jik} - f_{ijk} \tag{7}$$

transforms *homogeneously*. Thus the Lagrangian

$$\mathscr{L} = (-g)^{1/2}[\tfrac{1}{2}\mathrm{i}(\bar{\psi}\gamma^i D_i\psi - \overline{D_i\psi}\gamma^i\psi) - m\bar{\psi}\psi + (\alpha/8)\epsilon^{ijkl} T_{ijk}\bar{\psi}\gamma_l\gamma_5\psi + (1/\kappa^2)R_{ij}{}^{ij}] \tag{8}$$

is local Lorentz invariant (and thus generally invariant) independent of the scalar parameter α that determines the coupling strength of torsion to the matter field. Clearly from the Lagrangian (8) one has

$$C_{ijk} = C_{ijk}^{(0)} + C_{ijk}^{(2)} \tag{9}$$

where now the contortion $C_{ijk}^{(2)}$ is given by

$$C_{ijk}^{(2)} = \tfrac{1}{8}\kappa^2(1-\alpha)\epsilon_{ijkl}\bar{\psi}\gamma^l\gamma^5\psi. \tag{10}$$

Now one immediately notices that the theory reduces to Einstein's when $\alpha = 1$ and to Cartan's when $\alpha = 0$. Thus the spinor interaction with gravitation admits a more general coupling described by the Lagrangian (8) which yields both Einstein's and Cartan's theories as special cases. Remember that in Cartan's theory only the affine connection (or equivalently the gauge potentials of the Lorentz group) coupled to the matter field. However, we would like to emphasise that there is no *a priori* reason why torsion itself should not couple to the matter field directly. In some sense in a space–time with torsion, it would be more natural to expect a direct coupling between torsion and matter field. Indeed, as we have seen torsion does have a direct coupling to matter field in the gauge formalism of Einstein's theory with the unique strength (i.e. with $\alpha = 1$). The fact that the geometry allows the additional coupling constant comes from the fact that the Poincaré group which is the holomony group of the geometry has *two* Casimir invariants, mass and spin angular momentum. Clearly the new parameter

α determines the coupling strength of the torsion to the spin density whereas the old Newton's constant has to do with the mass density of the source field. *In this respect the theory is indeed a natural generalisation of Einstein's theory.*

At this point one might wonder whether the arbitrary parameter could be related to the coupling constant of the Lorentz gauge group. However, that cannot be the case since one can easily show (Cho 1976b) that the gauge formalism of Cartan's theory does not admit an arbitrary coupling for the gauge group other than the gravitational constant.

It has been speculated by Trautman (1973) that the repulsive spin–spin interaction in Cartan's theory might prevent singularities of Einstein's theory†. The nature of the interaction in our theory can easily be figured out. Substituting equation (9) into the Lagrangian (7) one obtains

$$\mathscr{L} = (-g)^{1/2}[(i/2)(\bar{\psi}\gamma^i\,\partial_i\psi - \partial_i\bar{\psi}\gamma^i\psi) - m\bar{\psi}\psi - \tfrac{1}{8}\epsilon^{ijkl}f_{ijk}\bar{\psi}\gamma_l\gamma_5\psi + \tfrac{3}{32}\kappa^2(1-\alpha)^2(\bar{\psi}\gamma^i\gamma^5\psi)^2 + \kappa^{-2}R_E]. \qquad (7')$$

Thus the spin–spin interaction remains to be repulsive, but the coupling strength is left arbitrary. This means that the size of the contracted universe will now depend on α. In particular it can be pointlike (yielding the singularity) in the Einstein limit. The arbitrary coupling strength of the torsion to matter field in our theory must be compared to the gravitational constant that determines the coupling strength of the metric to matter field, both of which could only be determined by experiment.

Finally we observe that although in the above example the simplest spinor matter field is considered it becomes clear how one can generalise the theory in the presence of an arbitrary source field: simply add the lowest order possible or the 'minimal' coupling of torsion to the spin density of the source field with an arbitrary strength‡. In view of the recent revived interest in the theory of gravitation with torsion in connection with supergravity, the new degree of freedom of the theory seems worth studying and might be relevant, for instance, to improve the renormalisability of the theory. The detailed dynamical properties of the theory will be discussed elsewhere.

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† Recently a contrary remark has been made on the issue (Kerlick 1975, Heyl *et al* 1976) based on the non-negative energy density assumption. However, to settle the issue completely one has to take into account quantum effects, in which case the non-negative energy density condition may not be respected. See e.g. Donald (1978) for more discussion.

‡ If the spin density is not totally antisymmetric as may be the case when fields of higher spin are present, one may either have to introduce more than one spin–torsion coupling constant, or use a properly symmetrised coupling to keep only one constant.

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